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LETTER TO THE EDITOR

AC response near percolation threshold: transfer matrix calculation in 2D

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Abstract. The complex admittivity of a network of resistors and capacitors is calculated at p_c on a square lattice using a transfer matrix. The loss tangent shows a region of anomalous frequency scaling, with $t = s$. Finite-size scaling reveals two crossover frequencies, as expected from duality.

This letter describes an application of the transfer matrix method to calculating the complex impedance of an electrical network near its percolation threshold. A transfer matrix was previously used to calculate the static (DC) conductance of a resistor network by Derrida and Vannimenus (1982). Subsequent applications followed (Herrmann *et al* 1984, Zabolitzky 1984, Saleur and Derrida 1985). These determined the 'superconductivity' (Straley 1977) or dielectric constant exponent, s, and the geometrical exponents γ and ν for lattices in 2D and 3D. In all of these studies, the electrical network consisted of resistive elements chosen from a bimodal distribution. This letter treats a network composed of bonds which are either purely resistive or purely capacitative. Thus the AC response of this lattice models the dielectric response of a composite medium in which conducting elements have been embedded in an insulator.

Anomalous behaviour of the AC conductivity near percolation threshold is a subject of recent interest; in contrast, anomalous DC conductivity as $p \to p_c$ has long been recognised. Above p_c, the conductivity of a system of finite (probability p) and infinite (probability $1-p$) resistances scales as $\Sigma \sim (p-p_c)^t$ (Kirkpatrick 1971, Last and Thouless 1971, Webman *et al* 1975). A homogeneous function representation of the conductivity of a mixture with conductivities σ_1 and σ_2 was previously suggested by Straley (Straley 1976, Webman *et al* 1977a). With $f_\pm$ a scaling function which has different asymptotic values above and below p_c for different extremes of its argument this relation reads

$$\Sigma = \sigma_1 |p-p_c|^t f_\pm[(\sigma_2/\sigma_1)|p-p_c|^{-(t+s)}].$$

Among the many consequences of this relation is that, upon analytic continuation of σ_k, the static dielectric constant of the composite is seen to scale as $\varepsilon \sim |p-p_c|^{-s}$ (Efros and Shklovskii 1976, Webman *et al* 1977b, Stroud and Bergman 1982). Straley's relation provides predictions concerning the system's response to an AC source if we explicitly

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substitute $\sigma_k \to \sigma_k + i\omega C_k$. In the simple case of a resistor/capacitor mixture $\sigma_1 \to \sigma_0$ and $\sigma_2 \to i\omega C_0$ and one writes the complex admittivity, Σ, as (Luck 1985)

$$\Sigma = \sigma_0 |p - p_c|^t f_\pm[(i\omega/\omega_0)|p - p_c|^{-(t+s)}] \tag{1}$$

when $p \to p_c^\pm$ and $\omega \ll \omega_0$, where $\omega_0 \equiv \sigma_0/C_0$. Equation (1) will produce the conventional limiting forms discussed above:

$$\Sigma \sim \sigma_0(p - p_c)^t \qquad \Sigma \sim \sigma_0(p_c - p)^{-s} \qquad \text{for } p \to p_c^+,\ p_c^-, \text{ respectively}$$

$$\varepsilon \sim C_0|p - p_c|^{-s} \qquad \text{for } p \to p_c^\pm$$

for frequencies much less than the crossover frequency $\omega^* \equiv \omega_0|p - p_c|^{(t+s)}$. For $\omega^* \ll \omega \ll \omega_0$ one has an anomalous scaling of the admittivity with frequency:

$$\Sigma \sim \sigma_0\, e^{iu\pi/2}(\omega/\omega_0)^u \tag{2}$$

where $u \equiv t/t+s$. Equation (2) immediately suggests that anomalous scaling may be observable in the loss tangent (Clerc *et al* 1984). It has also been predicted in the noise properties (Rammal 1984) of a system near p_c.

Experimental evidence for anomalous scaling of the complex admittance with frequency has recently been presented (Leibowitz and Gefen 1984, Bhattacharya *et al* 1985, van Dijk 1985, Niklasson and Granqvist 1985). It should be noted that only in the case of Niklasson and Granqvist did the measured values of the exponents conform to theory for the response of a continuous ohmic dielectric medium embedded in a given dimensionality. (Recent values of exponents for AC response and noise are summarised in Luck (1985).)

Our method is a simple extension of the method described in Derrida and Vannimenus (1982). As is drawn in their figure 1, one generates a strip of arbitrary length and finite width, $M+1$ say, by adding strips of unit length and width $M+1$ to the edge of an existing strip. Each new strip consists of a set of $M+1$ horizontal and M vertical bonds. The admittance of each bond is 1 with probability p and $i\omega$ with probability $1-p$. The new strip is added so as to continue the construction of a square lattice from the existing strip. The top and bottom horizontal elements of the new strip are always given zero resistance. They form a superconducting path across the strip and thus one calculates a line to line admittance with this method. If one applies external currents only at the nodes at the growing edge of the strip and labels these nodes with a vector index i with $1 \leq i \leq M+1$, then the voltages at these points are linearly related to the currents via an $(M+1)\times(M+1)$ matrix:

$$\begin{pmatrix} V_1 \\ \vdots \\ V_{M+1} \end{pmatrix} = \tilde{A}_N \begin{pmatrix} I_1 \\ \vdots \\ I_{M+1} \end{pmatrix}. \tag{3}$$

The basic idea behind the transfer matrix is that if $\tilde{A}_N$ is known for a strip of length N, $\tilde{A}_{N+1}$ can be calculated exactly in terms of $\tilde{A}_N$ and the $2M+1$ new circuit elements. If the ith new horizontal resistor has conductance h_i and the ith new vertical resistor has conductance g_i, then the recursion relation is

$$\tilde{A}_{N+1} = [(\tilde{h}^{-1} + \tilde{A}_N)^{-1} + \tilde{G}]^{-1} \tag{4}$$

where

$$\tilde{h}_{ij}^{-1} = 1/h_i\delta_{ij}$$

$$\tilde{G}_{ij} = -g_{i-1}\delta_{i-1,j} + (g_{i-1} + g_i)\delta_{ij} - g_i\delta_{i+1,j}.$$

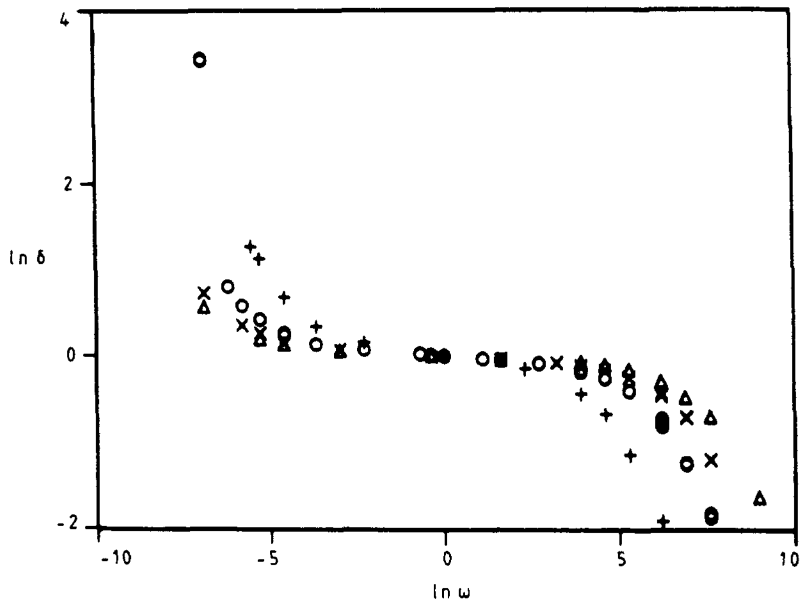

Figure 1. Log–log plot of loss tangent ($\mathrm{Re}\,\Sigma/\mathrm{Im}\,\Sigma$) against frequency for strips of various widths M at p_c. In a range of frequencies centred about $\omega_0 = 1$ the loss tangent becomes frequency independent. Its value in that range is unity, which implies that $t = s$. Widths: +, 9; ○, 19; ×, 29; △, 39.

If a current I is applied at the $(M+1)$th node and removed from the first node, then Ohm's law reads

$$\Sigma_N(V_{M+1} - V_1) = I \tag{5}$$

where Σ_N is the admittance of the strip of length N. Substituting $I_{M+1} = -I_1 = I$, and all other $I_k \equiv 0$ in equation (3), we find that the complex admittance per unit length of the strip is simply

$$\Sigma_N/N = N^{-1}[(\tilde{A}_N)_{mm} + (\tilde{A}_N)_{11} - (\tilde{A}_N)_{m1} - (\tilde{A}_N)_{1m}]^{-1}. \tag{6}$$

This quantity, multiplied by the width M, is expected to converge to the admittance of a nework of $M \times M$ bonds in the limit as $N \to \infty$. (In 2D, admittance and admittivity are equivalent.)

In the limit $\omega = 0$, there is no imaginary component of the admittivity and we calculate the DC conductivity for unit resistors mixed with infinite resistors at p_c. The calculation is carried out precisely at $p_c = \frac{1}{2}$ and finite-size scaling arguments determine the exponent t/ν via (Mitescu *et al* 1982)

$$\Sigma \sim M^{-t/\nu}. \tag{7}$$

That is, the width of the system determines an effective value of $(p - p_c)$ which is non-zero and scales as $M^{-1/\nu}$. Critical quantities (such as ω^*) are determined with respect to this effective distance from the critical point. Strips of width $M+1 = 8$, 10, 20, 30 and 40 with lengths of order 10^5–10^6 were generated. These lengths were sufficient to give asymptotic values of Σ_N/N; this was also the range of strip lengths found to be sufficient by Derrida and Vannimenus (1982). Our results for Σ were found to agree with those of their figure 2 to within the cited accuracy; we find that $t/\nu \simeq 0.96 \pm 0.01$.

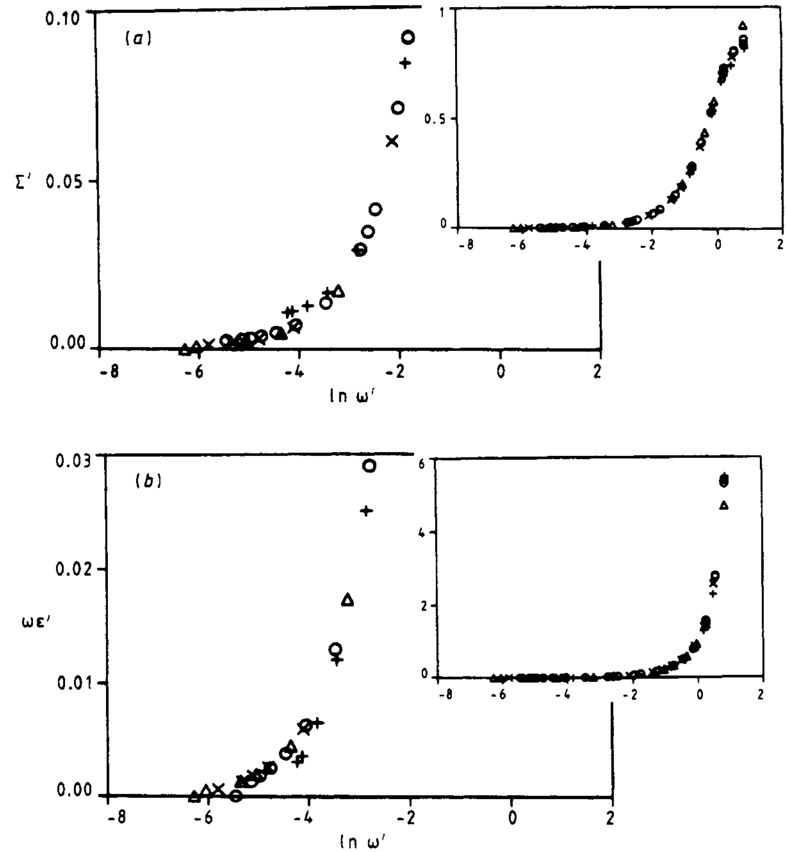

Figure 2. (a) Scaled conductivity, $\Sigma' \equiv \Sigma M^{-s/\nu}$, against logarithm of the scaled frequency, $\omega' \equiv \omega / M^{(t+s)/\nu}$. ($b$) Scaled dielectric constant, $\omega\varepsilon' \equiv \omega\varepsilon M^{-s/\nu}$, against logarithm of scaled frequency as in (a). Scaled variables are suggested by equation (9) with $t = s = 1.3$. Universal behaviour should hold for $\omega \gg \omega^*$ (see text). Smaller insets are complete data sets. Larger plots are small frequency data with expanded vertical scale to show the breakdown of scaling near ω^* (for widths see figure 1).

Data for the response to a non-zero driving frequency are displayed in figure 1. The loss tangent, $\mathrm{Re}\,\Sigma/\mathrm{Im}\,\Sigma \equiv \tan\delta$, where $\delta(\omega)$ is the loss angle. Equation (2) predicts that the loss angle becomes frequency independent in a critical range between ω^* and ω_0 where it takes on the value

$$\delta_c = \pi s / 2(t+s). \qquad (8)$$

For our system, ω_0 is unity. In 2D, the well known duality relation $t = s$ (Straley 1977) implies that $\delta_c = \pi/4$. Thus the loss tangent should become unity in the critical range; figure 1 displays this behaviour.

Beyond the critical range, there must exist crossovers to values of the loss tangent which are dictated by analyticity properties of the complex admittivity. For high frequencies (Luck 1985) this behaviour is $\tan\delta \sim \omega^{-1}$ if the capacitative phase percolates and $\tan\delta \sim \omega$ if it does not. The infinite square bond lattice with $p_c = \frac{1}{2}$ is a special case in which a special high frequency crossover is expected. A very high frequency crossover scale $\omega^{**} \sim \omega_0 |p - p_c|^{-(t+s)}$ is set by duality, which demands that $\tan\delta(\omega) =$

$\tan^{-1}\delta(1/\omega)$. This self-dual lattice really demands a scaling form for the admittance which includes a two-argument scaling function: $g(\omega/\omega^*, \omega/\omega^{**})$. However, $\omega^* \sim 1/\omega^{**}$ and their difference diverges as $p \to p_c$. Thus we can split the scaling law for Σ into two laws; each one holds in the vicinity of its characteristic crossover frequency.

Figure 2 displays the real and imaginary parts of the admittivity scaled according to the law

$$\Sigma \sim M^{s/\nu} g_1[\omega/M^{(t+s)/\nu}]. \quad (9)$$

By finite-size scaling $M \sim (p-p_c)^{-\nu}$, so data for various widths should fall on the same universal curve above and below the crossover frequency ω^{**}. This is borne out well in figure 2; equation (9) describes the critical region and the vicinity of the high frequency crossover ω^{**}. Further, equation (9) must fail in the vicinity of ω^*. For these frequencies, therefore, the data of figure 2 should and do leave the scaling curve. The crossover frequency at which this occurs is ω^*; this is checked by plotting the data in figure 3, as discussed in the next paragraph.

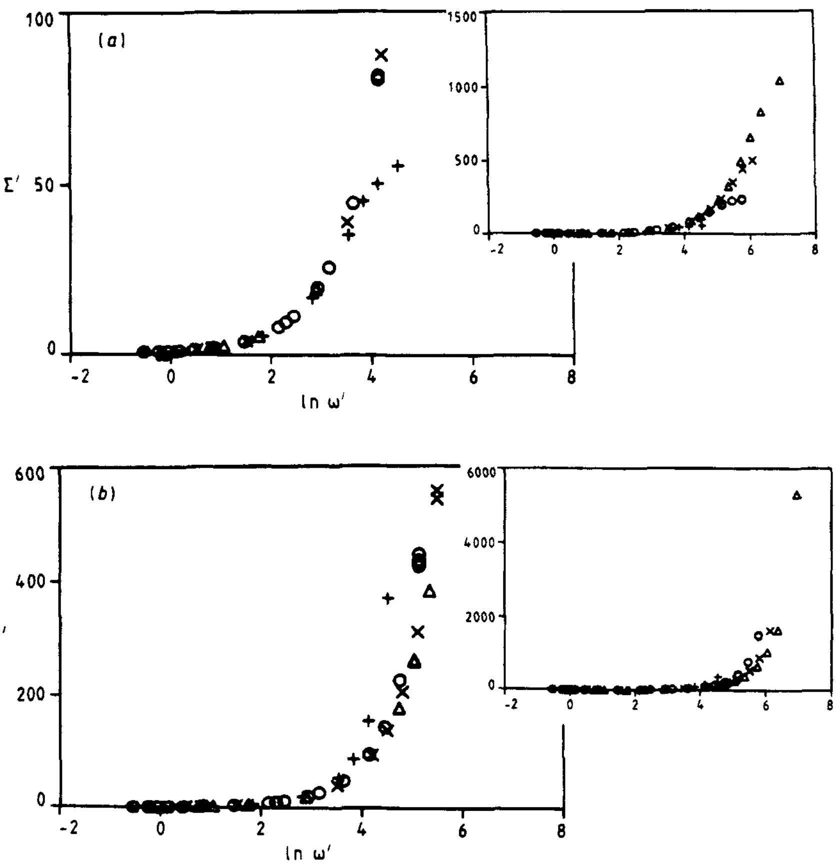

Figure 3. (a) Scaled conductivity, $\Sigma' \equiv \Sigma M^{t/\nu}$, against logarithm of the scaled frequency, $\omega' \equiv \omega M^{(t+s)/\nu}$. ($b$) Scaled dielectric constant, $\omega\varepsilon' \equiv \omega\varepsilon M^{t/\nu}$, against logarithm of scaled frequency as in (a). Scaled variables are suggested by equation (10) with $t = s = 1.3$. Universal behaviour should hold for $\omega \ll \omega^{**}$ (see text). Smaller insets are complete data sets. Larger plots are small frequency data with expanded vertical scale to show the onset of scaling behaviour far below ω^{**} (for widths see figure 1).

At low frequencies, the admittance of the capacitors goes to zero with ω, and the behaviour of tan δ depends on whether or not the system is above or below p_c. One expects the loss tangent to vanish as ω if $p < p_c$, and to diverge as ω^{-1} if $p > p_c$ for $\omega \ll \omega^*$. In fact, since our calculation takes place on a finite lattice, one is always above p_c for the resistors and thus one expects the latter behaviour. This low frequency behaviour is seen in figure 1. The scaling law dual to equation (9) which successfully describes this crossover is

$$\Sigma \sim M^{-t/\nu} g_2[\omega M^{(t+s)/\nu}]. \tag{10}$$

Admittance data scaled according to equation (10) are plotted in figure 3. As claimed, the data scale well for frequencies below the lower crossover frequency ω^* and above it in the critical regime. For frequencies above ω^{**}, equation (10) fails as expected, and scaling behaviour is described by equation (9) and figure 2. The crossover frequency at which this occurs is ω^{**}. This is confirmed by the adherence of the data of figure 2 to a universal scaling function for frequencies far above ω^*.

In conclusion, we have used a transfer matrix method to extract the anomalous frequency dependence of the AC response of a 2D network of conductors and insulators near p_c. The results confirm the predictions of the traditional scaling theory.

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Note added. After this manuscript was submitted, we were made aware of a preprint by J M Laugier, J P Clerc, G Giraud and J M Luck, which also treats the AC response of a percolation model in two dimensions via a transfer matrix.

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